

## AN EXAMPLE OF NON-EMBEDDABILITY OF THE RICCI FLOW

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ABSTRACT. For an evolution of metrics  $(M, g_t)$  there is a  $t$ -smooth family of embeddings  $e_t : M \rightarrow \mathbb{R}^N$  inducing  $g_t$ , but in general there is no family of embeddings extending a given initial embedding  $e_0$ . We give an example of this phenomenon when  $g_t$  is the evolution of  $g_0$  under the Ricci flow. We show that there are embeddings  $e_0$  inducing  $g_0$  which do not admit of  $t$ -smooth extensions to  $e_t$  inducing  $g_t$  for any  $t > 0$ . We also find hypersurfaces of  $\dim > 2$  that will not remain a hypersurface under Ricci flow for any positive time.

## 1. PRELIMINARIES

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function and let

$$M^n = \text{graph}(f) \hookrightarrow \mathbb{R}^{n+1}$$

Let  $\langle \cdot, \cdot \rangle$  be the standard inner product on  $\mathbb{R}^{n+1}$ , and  $g$  be the induced metric on  $M$  from  $\mathbb{R}^{n+1}$ . Let  $\nabla, D$  be the standard covariant derivatives on  $\mathbb{R}^n, \mathbb{R}^{n+1}$  respectively. First we will compute the metric and curvature of  $M$ . Note that  $M$  is diffeomorphic to  $\mathbb{R}^n$  and we can cover it by one chart, which we will do from now on. Now

$$F = (\text{id}, f) : M \rightarrow \mathbb{R}^{n+1}$$

is an embedding, so the tangent vectors to  $M$  are  $\partial_i F = (e_i, \partial_i f)$  where  $e_i$ 's are the standard basis of  $\mathbb{R}^n$ . The components of  $g$  are

$$(1.1) \quad g_{ij} = \langle \partial_i F, \partial_j F \rangle = \delta_{ij} + \partial_i f \partial_j f$$

The unit normal to  $M$  is

$$(1.2) \quad N = \frac{1}{\sqrt{1 + |\nabla f|^2}} (\nabla f, -1)$$

Also the components of the second fundamental form are

$$h_{ij} = -\langle D_{\partial_i F} \partial_j F, N \rangle$$

Since

$$D_{\partial_i F} \partial_j F = \sum_k \partial_i F^k \partial_k \partial_j F = \partial_i \partial_j F + \partial_i f \partial_{n+1} \partial_j F = \partial_i \partial_j F = (0, \partial_i \partial_j f)$$

(because  $\partial_j F$  is independent of  $x_{n+1}$ ) we have

$$(1.3) \quad h_{ij} = -\langle \partial_i \partial_j F, N \rangle = \frac{\partial_i \partial_j f}{\sqrt{1 + |\nabla f|^2}}$$

The Gauss equation implies

$$(1.4) \quad R_{ijkl} = h_{il} h_{jk} - h_{ik} h_{jl} = \frac{1}{1 + |\nabla f|^2} (\partial_i \partial_l f \partial_j \partial_k f - \partial_i \partial_k f \partial_j \partial_l f)$$

By an easy induction on  $n$  we find

$$(1.5) \quad \det g_{ij} = 1 + \sum (\partial_i f)^2 = 1 + |\nabla f|^2$$

and also the components of  $g^{-1}$

$$(1.6) \quad g^{ij} = \delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla f|^2}$$

Now we can compute the Christoffel symbols

$$(1.7) \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = \frac{\partial_k f \partial_i \partial_j f}{1 + |\nabla f|^2}$$

## 2. THE EXAMPLE

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be given by

$$(2.1) \quad f(x_1, \dots, x_n) = \sum_{r,q} a_{rq} x_r x_q^2$$

(where  $(a_{rq})$  is not necessarily symmetric) We are interested in the evolution of  $(M, g)$  under the Ricci flow. Let  $p$  denote the origin in  $\mathbb{R}^{n+1}$ . The derivatives of  $f$  are

$$(2.2) \quad \partial_i f = \sum_{r \neq i} a_{ri} x_r x_i + \sum_{q \neq i} a_{iq} x_q^2 + 3a_{ii} x_i^2$$

$$(2.3) \quad \partial_i \partial_j f = \begin{cases} 2a_{ij} x_j + 2a_{ji} x_i & i \neq j \\ \sum_{q \neq i} 2a_{qi} x_q + 6a_{ii} x_i & i = j \end{cases}$$

Note that all these expressions vanish at the origin, so both the curvature and the connection vanish at  $p$ . In addition we have  $g_{ij} = \delta_{ij}$  at  $p$ .

We know that under the Ricci flow the Riemann curvature tensor evolves as

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm + Rm * Ric$$

where  $A * B$  is a sum of contractions of components of the tensors  $A$  and  $B$  by the metric. Now if we look at this equation at  $x = p$  and  $t = 0$  we get

$$(2.4) \quad \left. \frac{\partial}{\partial t} Rm \right|_{t=0} = \Delta Rm$$

But

$$\Delta Rm = g^{ij} \partial_i \partial_j Rm + \Gamma * \partial Rm + \partial \Gamma * Rm + \Gamma * Rm$$

Since  $\Gamma$  and  $Rm$  vanish at  $p$  and  $g$  is the identity there, we obtain

$$(2.5) \quad \left. \frac{\partial}{\partial t} Rm \right|_{t=0} = \sum \partial_i \partial_i Rm$$

at  $x = p$ . We define the tensor  $A$  by

$$(2.6) \quad A_{ijkl} := \partial_i \partial_l f \partial_j \partial_k f - \partial_i \partial_k f \partial_j \partial_l f$$

then  $Rm = \frac{1}{1 + |\nabla f|^2} A$ , and as  $A, \partial A$  vanish at  $p$ , we have

$$(2.7) \quad \partial \partial Rm = \frac{1}{1 + |\nabla f|^2} \partial \partial A$$

at  $p$ . Since  $A$  has the symmetries of the curvature tensor, it is enough to compute  $A_{ijkl}$  for  $i < j, k < l, i \leq k$

$$A_{ijkl} = \begin{cases} \begin{aligned} &4(a_{il}x_l + a_{li}x_i)(a_{kj}x_j + a_{jk}x_k) \\ &-4(a_{ik}x_k + a_{ki}x_i)(a_{jl}x_l + a_{lj}x_j) \end{aligned} & i, j, k, l \text{ are all distinct} \\ \begin{aligned} &4(a_{il}x_l + a_{li}x_i)(\sum_{q \neq j} a_{qj}x_q + 3a_{jj}x_j) \\ &-4(a_{ij}x_j + a_{ji}x_i)(a_{jl}x_l + a_{lj}x_j) \end{aligned} & i < k = j < l \\ \begin{aligned} &4(a_{ij}x_j + a_{ji}x_i)(a_{kj}x_j + a_{jk}x_k) \\ &-4(a_{ik}x_k + a_{ki}x_i)(\sum_{q \neq j} a_{qj}x_q + 3a_{jj}x_j) \end{aligned} & i < k < l = j \\ \begin{aligned} &4(a_{il}x_l + a_{li}x_i)(a_{ij}x_j + a_{ji}x_i) \\ &-4(a_{jl}x_l + a_{lj}x_j)(\sum_{q \neq i} a_{qi}x_q + 3a_{ii}x_i) \end{aligned} & i = k, j \neq l \\ \begin{aligned} &4(a_{ij}x_j + a_{ji}x_i)^2 \\ &-4(\sum_{q \neq i} a_{qi}x_q + 3a_{ii}x_i)(\sum_{r \neq j} a_{rj}x_r + 3a_{jj}x_j) \end{aligned} & i = k, j = l \end{cases}$$

Therefore

$$\sum \partial_s \partial_s A_{ijkl} = \begin{cases} 0 & i, j, k, l \\ & \text{are all distinct} \\ 8(a_{il}a_{lj} + a_{li}a_{ji}) - 8a_{ij}a_{lj} & i < k = j < l \\ 8a_{ij}a_{kj} - 8(a_{ik}a_{kj} + a_{ki}a_{ij}) & i < k < l = j \\ 8a_{li}a_{ji} - 8(a_{jl}a_{li} + a_{lj}a_{ji}) & i = k, j \neq l \\ 8(a_{ij}^2 + a_{ji}^2) - 8(\sum_{q \neq i, j} a_{qi}a_{qj} + 3a_{ii}a_{ij} + 3a_{jj}a_{ji}) & i = k, j = l \end{cases}$$

Now if we choose  $(a_{rq})$  such that

$$(2.8) \quad a_{\alpha\beta}a_{\beta\gamma} + a_{\beta\alpha}a_{\alpha\gamma} = a_{\alpha\gamma}a_{\beta\gamma}$$

for  $\alpha, \beta, \gamma$  mutually distinct then all non-diagonal entries of  $\sum \partial_s \partial_s A_{ijkl}$  and hence  $\frac{\partial}{\partial t} Rm$  vanish at  $x = p$  and  $t = 0$  (note that  $\frac{1}{1+|\nabla f|^2} = 1$  at  $p$ ).

Let  $a_{ij} = 0$  for  $i < j$  and  $a_{ji} = 1$ , then (2.8) holds. Also let the diagonal elements  $a_{ii}$  to be 1, then the diagonal entries of  $\frac{\partial}{\partial t} Rm(0, p)$  are all negative. Therefore as  $Rm(0, p) = 0$ , the sectional curvatures of  $Rm(t, p)$  will be negative for small  $t$ . Therefore for any positive  $t$ ,  $(M, g(t))$  is no longer a hypersurface when  $n \geq 3$ . Note that no neighborhood of  $p$  can be embedded in  $\mathbb{R}^{n+1}$  for any  $t > 0$  and using this we can construct closed hypersurfaces that will not remain a hypersurface for any positive time under Ricci flow.

We also observe that for  $n \geq 2$  if we consider the embedding

$$\varphi : M^n \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+k}$$

for any  $k \geq 1$ , then there is no evolution of  $\varphi$  that induces the Ricci flow on  $M$ . In fact, if such evolution of  $\varphi$  exists, then by the Gauss equation we will have

$$Rm(t)(X, Y, Z, W) = \langle \Pi(t)(X, W), \Pi(t)(Y, Z) \rangle - \langle \Pi(t)(X, Z), \Pi(t)(Y, W) \rangle$$

where  $\Pi$  is the second fundamental form. Differentiating we obtain

$$\begin{aligned} \frac{\partial}{\partial t} Rm(0)(X, Y, Z, W) &= \left\langle \frac{\partial}{\partial t} \Pi(0)(X, W), \Pi(0)(Y, Z) \right\rangle \\ &\quad + \langle \Pi(0)(X, W), \frac{\partial}{\partial t} \Pi(0)(Y, Z) \rangle \\ &\quad - \left\langle \frac{\partial}{\partial t} \Pi(0)(X, Z), \Pi(0)(Y, W) \right\rangle \\ &\quad - \langle \Pi(0)(X, Z), \frac{\partial}{\partial t} \Pi(0)(Y, W) \rangle \end{aligned}$$

at  $p$ . But  $\Pi(0) = 0$  at  $p$  and this contradicts the fact that  $\frac{\partial}{\partial t} Rm(0) \neq 0$ .

Also note that even changing the metric on  $\mathbb{R}^{n+k}$  will not allow the existence of an evolution of  $\varphi$  that induces  $g(t)$  since in this case

$$Rm(t)(X, Y, Z, W) = \eta(t)(\Pi(t)(X, W), \Pi(t)(Y, Z)) - \eta(t)(\Pi(t)(X, Z), \Pi(t)(Y, W))$$

where  $\eta(t)$  is the evolution of the standard metric on  $\mathbb{R}^{n+k}$ . Thus

$$\begin{aligned} \frac{\partial}{\partial t} Rm(0)(X, Y, Z, W) &= \left\langle \frac{\partial}{\partial t} \Pi(0)(X, W), \Pi(0)(Y, Z) \right\rangle \\ &\quad + \langle \Pi(0)(X, W), \frac{\partial}{\partial t} \Pi(0)(Y, Z) \rangle \\ &\quad - \left\langle \frac{\partial}{\partial t} \Pi(0)(X, Z), \Pi(0)(Y, W) \right\rangle \\ &\quad - \langle \Pi(0)(X, Z), \frac{\partial}{\partial t} \Pi(0)(Y, W) \rangle \\ &\quad + \frac{\partial}{\partial t} \eta(0)(\Pi(0)(X, Z), \Pi(0)(Y, W)) \end{aligned}$$

at  $p$ . Again  $\Pi(0) = 0$  at  $p$  and we get a contradiction with  $\frac{\partial}{\partial t} Rm(0) \neq 0$ .

*Remark.* For a generic isometric embedding of a Riemannian manifold  $(M, g)$  in  $\mathbb{R}^N$ , the metric  $g(t)$  can be embedded in  $\mathbb{R}^N$  for small  $t > 0$ . The problem is that we do not know if and when the evolution of the metric encounters an obstacle beyond which we may not be able to extend the embedding. A successful resolution of this problem will have interesting consequences. For example it may allow us to obtain isometric embeddings of a surface of genus  $> 1$  with constant negative curvature in  $\mathbb{R}^5$ .

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